Derivation of a Squared Ellipsoidal Lobe Function

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1 Squared Spheroidal Lobe (SSL)

To derive a squared ellipsoidal lobe (SEL) function, we start from the following squared spheroidal lobe (SSL):

$$\pi\acute{\alpha}^2 D\left(\cos\frac{\theta}{2}, \acute{\alpha}\right) = \frac{4\acute{\alpha}^4}{\left(1-\cos\theta+\acute{\alpha}^2(1+\cos\theta)\right)^2}.$$

where θ is the angle between a direction $\boldsymbol{\omega} \in \mathrm{S}^2$ and the lobe axis $\boldsymbol{\omega}_z \in \mathrm{S}^2$, $\acute{\alpha} \in [0,1]$ is the roughness of the lobe, and $D(\cos\theta, \acute{\alpha})$ is the isotropic GGX distribution [TR75, WMLT07]. Tokuyoshi and Harada [TH17] derived $\sqrt{\pi D\left(\cos\frac{\theta}{2}, \acute{\alpha}\right)}$ is a spheroid whose center and semiaxes in the lobe space are $\left[0, 0, \frac{1-\acute{\alpha}^2}{2\acute{\alpha}}\right]$ and $\left[1, 1, \frac{1+\acute{\alpha}^2}{2\acute{\alpha}}\right]$, respectively. Therefore, the lobe-space center $\mathbf{c}$ and semiaxes $\mathbf{r}$ of $\sqrt{\pi\acute{\alpha}^2 D\left(\cos\frac{\theta}{2}, \acute{\alpha}\right)}$ are given by

$$\mathbf{c} = \left[0, 0, \frac{1-\acute{\alpha}^2}{2}\right],$$

$$\mathbf{r} = \left[\acute{\alpha}, \acute{\alpha}, \frac{1+\acute{\alpha}^2}{2}\right].$$

2 Extension to a Squared Ellipsoidal Lobe (SEL)

This paper extends semiaxes $\mathbf{r}$ using anisotropic roughness parameters $\left[\acute{\alpha}_x, \acute{\alpha}_y\right]$ as follows:

$$\mathbf{r} = \left[\acute{\alpha}_x, \acute{\alpha}_y, \frac{1+\acute{\alpha}_{\max}^2}{2}\right],$$

where $\acute{\alpha}_{\max} = \max\left(\acute{\alpha}_x, \acute{\alpha}_y\right)$. For this, the lobe-space center is

$$\mathbf{c} = \left[0, 0, \frac{1-\acute{\alpha}_{\max}^2}{2}\right].$$

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Our SEL function is given by the squared distance from the origin to this ellipsoid. Therefore, we derive the SEL using the intersection of this ellipsoid and a line. A position on this line is given by

$$\mathbf{p} = t\boldsymbol{\omega},$$

where t is a distance from the origin. The ellipsoid-line intersection is equivalently rewritten into the intersection of a transformed line and a unit sphere centered at the origin. For this, a position on this line is given by

$$\mathbf{p}' = (t\boldsymbol{\omega} - \mathbf{c})\begin{bmatrix} \frac{1}{\acute{\alpha}_x} & 0 & 0 \\ 0 & \frac{1}{\acute{\alpha}_y} & 0 \\ 0 & 0 & \frac{2}{1+\acute{\alpha}^2_{\max}} \end{bmatrix} = t\mathbf{d} + \mathbf{s},$$

$$\mathbf{d} = \left[\frac{x}{\acute{\alpha}_x}, \frac{y}{\acute{\alpha}_y}, \frac{2z}{1+\acute{\alpha}^2_{\max}}\right], \tag{1}$$

$$\mathbf{s} = \left[0, 0, -\frac{1-\acute{\alpha}^2_{\max}}{1+\acute{\alpha}^2_{\max}}\right]. \tag{2}$$

where $\boldsymbol{\omega} = [x, y, z]$. The intersection point of this line and the unit sphere is given as $\|\mathbf{p}'\|^2 = 1$. It is rewritten into a quadratic equation:

$$\|\mathbf{d}\|^2 t^2 + 2(\mathbf{d}\cdot\mathbf{s})t + \|\mathbf{s}\|^2 - 1 = 0.$$

The positive solution of this equation is given by

$$t = \frac{\sqrt{(\mathbf{d}\cdot\mathbf{s})^2 - \|\mathbf{d}\|^2(\|\mathbf{s}\|^2 - 1)} - \mathbf{d}\cdot\mathbf{s}}{\|\mathbf{d}\|^2}. \tag{3}$$

Substituting Eq. (1) and Eq. (2) into Eq. (3), the solution is obtained as follows:

$$\begin{aligned}
t &= 2\frac{\left(1+\acute{\alpha}^2_{\max}\right)\sqrt{\frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_x}x^2 + \frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_y}y^2 + z^2} + z\left(1-\acute{\alpha}^2_{\max}\right)}{\left(1+\acute{\alpha}^2_{\max}\right)^2\left(\frac{x^2}{\acute{\alpha}^2_x} + \frac{y^2}{\acute{\alpha}^2_y} + \frac{4z^2}{\left(1+\acute{\alpha}^2_{\max}\right)^2}\right)} \\
&= 2\acute{\alpha}^2_{\max}\frac{\left(1+\acute{\alpha}^2_{\max}\right)\sqrt{\frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_x}x^2 + \frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_y}y^2 + z^2} + z\left(1-\acute{\alpha}^2_{\max}\right)}{\left(1+\acute{\alpha}^2_{\max}\right)^2\left(\frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_x}x^2 + \frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_y}y^2\right) + 4z^2\acute{\alpha}^2_{\max}} \\
&= 2\acute{\alpha}^2_{\max}\frac{\left(1+\acute{\alpha}^2_{\max}\right)\sqrt{\frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_x}x^2 + \frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_y}y^2 + z^2} + z\left(1-\acute{\alpha}^2_{\max}\right)}{\left(1+\acute{\alpha}^2_{\max}\right)^2\left(\frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_x}x^2 + \frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_y}y^2 + z^2\right) - z^2\left(1+\acute{\alpha}^2_{\max}\right)^2 + 4z^2\acute{\alpha}^2_{\max}} \\
&= 2\acute{\alpha}^2_{\max}\frac{\left(1+\acute{\alpha}^2_{\max}\right)\sqrt{\frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_x}x^2 + \frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_y}y^2 + z^2} + z\left(1-\acute{\alpha}^2_{\max}\right)}{\left(1+\acute{\alpha}^2_{\max}\right)^2\left(\frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_x}x^2 + \frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_y}y^2 + z^2\right) - z^2\left(1-\acute{\alpha}^2_{\max}\right)^2} \\
&= \frac{2\acute{\alpha}^2_{\max}}{\left(1+\acute{\alpha}^2_{\max}\right)\sqrt{\frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_x}x^2 + \frac{\acute{\alpha}^2_{\max}}{\acute{\alpha}^2_y}y^2 + z^2} - z\left(1-\acute{\alpha}^2_{\max}\right)}.
\end{aligned}$$

Therefore, the SEL is derived as

$$K(\boldsymbol{\omega}; \mathbf{E}, \acute{\alpha}_x, \acute{\alpha}) = t^2 = \frac{4\acute{\alpha}_{\max}^4}{\left(\left(1+\acute{\alpha}_{\max}^2\right)\sqrt{\frac{\acute{\alpha}_{\max}^2}{\acute{\alpha}_x^2}x^2 + \frac{\acute{\alpha}_{\max}^2}{\acute{\alpha}_y^2}y^2 + z^2} - z\left(1-\acute{\alpha}_{\max}^2\right)\right)^2}.$$

where $\mathbf{E}$ is the 3×3 identity matrix. To represent the orientation of the lobe, Eq. (2) is generalized using a 3×3 orthogonal matrix $\mathbf{Q}$ as follows:

$$K(\boldsymbol{\omega}; \mathbf{Q}, \acute{\alpha}_x, \acute{\alpha}) = \frac{4\acute{\alpha}_{\max}^4}{\left(\left(1+\acute{\alpha}_{\max}^2\right)\sqrt{\frac{\acute{\alpha}_{\max}^2}{\acute{\alpha}_x^2}v_x^2 + \frac{\acute{\alpha}_{\max}^2}{\acute{\alpha}_y^2}v_y^2 + v_z^2} - v_z\left(1-\acute{\alpha}_{\max}^2\right)\right)^2}.$$

where $[v_x, v_y, v_z]^{\mathrm{T}} = \mathbf{Q}\boldsymbol{\omega}^{\mathrm{T}}$ is the direction transformed into the lobe space. This SEL can also be rewritten into the following form:

$$K(\boldsymbol{\omega}; \mathbf{Q}, \acute{\alpha}_x, \acute{\alpha}) = \frac{4\acute{\alpha}_{\max}^4}{\left((U - v_z) + \acute{\alpha}_{\max}^2 (U + v_z)\right)^2}.$$

where $U = \sqrt{\frac{\acute{\alpha}_{\max}^2}{\acute{\alpha}_x^2}v_x^2 + \frac{\acute{\alpha}_{\max}^2}{\acute{\alpha}_y^2}v_y^2 + v_z^2}$.

References

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